

3. SOLUTION OF POLYNOMIAL EQUATION BY BAIRSTOW-HITCHCOCK METHOD

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procedure

BAIRSTOW (n, a[], eps0, eps1, eps2, eps3, K) =:
(m, x[], y[], nat[], ex[]);

comment The Bairstow-Hitchcock iteration is used to find successively pairs of roots of a polynomial equation of degree n with coefficients a_i ($i = 0, 1, \dots, n$) where a_n is the constant term. On exit from the procedure, m is the number of pairs of roots found, x[i] and y[i] ($i = 1, \dots, m$) are a pair of real roots if nat[i]=1, the real and imaginary parts of a complex pair if nat[i]=-1, and ex[i] indicates which of the following conditions was met to exit from the iteration loop in finding this pair:

1. Remainders, r1, r0, become absolutely less than eps1.
2. Corrections, incrp, incrq, become absolutely less than eps2.
3. The ratios, incrp/p, incrq/q, become absolutely less than eps3.
4. The number of iterations becomes K.

In the last case, the pair of roots found is not reliable and no further effort to find additional roots is made. The quantity eps0 is used as a lower bound for the denominator in the expressions from which incrp and incrq are found.;

begin

integer (i, j, k, n1, n2, m1) ;
array (b, c[0 : n+1]) ;
BAIRSTOW **for** i := 0(1)n ; b_i := a_i
b_{n+1} := 0 ; n2 := entire((n+1)/2)
n1 := 2×n2
for m1 := 1(1)n2 ; **begin** p := 0 ; q := 0
for k := 1(1)K ; **begin**
for i := 0(1)n1 ; c_i := b_i
for j := n1-2, n1-4 ; **begin**
for i := 0(1)j ; **begin**
c_{i+1} := c_{i+1} - p × c_i
c_{i+2} := c_{i+2} - q × c_i **end end**
r0 := c_{n1} ; r1 := c_{n1-1}
s0 := c_{n1-2} ; s1 := c_{n1-3}
v0 := -q × s1 ; v1 := s0 - s1 × p
det0 := v1 × s0 - v0 × s1
if (abs(det0) < eps0) ; **begin**
p := p+1 ; q := q+1 ; **go to step end**
det1 := s0 × r1 - s1 × r0
det2 := r0 × v1 - v0 × s1
incrp := det1/det0 ; incrq := det2/det0
p := p + incrp ; q := q + incrq
if (abs(r0) < eps1) ; **begin**
if (abs(r1) < eps1) ; **begin**
ex_{m1} := 1 ; **go to next end end**
if (abs(incrp) < eps2) ; **begin**
if (abs(incrq) < eps2) ; **begin**

step:

ex_{m1} := 2 ; **go to next end end****if** (abs(incrp/p) < eps3) ; **begin****if** (abs(incrq/q) < eps3) ; **begin**ex_{m1} := 3 ; **go to next end end end**ex_{m1} := 4next: S := p/2 ; T := S² - q**if** (T ≥ 0) ; **begin** T := sqrt(T)nat_{m1} := 1 ; x_{m1} := S + Ty_{m1} := S - T **end****if** (T < 0) ; **begin** nat_{m1} := -1 ; x_{m1} := Sy_{m1} := sqrt(-T) **end****if** (ex_{m1} := 4) ; **go to out****for** j := 0(1) (n1-2) ; **begin**b_{j+1} := b_{j+1} - p × b_jb_{j+2} := b_{j+2} - q × b_j ; **end**n1 := n1 - 2 ; **if** (n1 < 1)out: **begin** m := m1 ; **return end****if** (n1 < 3) ; **begin**m1 := m1 + 1 ; ex_{m1} := 1p := b₁/b₀ ; q := b₂/b₀**go to next end****end end**

CERTIFICATION

3. SOLUTION OF POLYNOMIAL EQUATION BY BAIRSTOW-HITCHCOCK METHOD, A. A. Grau, *Communications ACM*, February, 1960.

Henry C. Thacher, Jr.,* Argonne National Laboratory, Argonne, Illinois.

Bairstow was coded for the Royal-Precision LGP-30 computer, using an interpretive floating point system (24.2) with 28 bits of significance. The translation from ALGOL was made by hand.

The following minor corrections were found necessary.

a. det 2 := r0 × v1 - v0 × s1 should be det 2 := r0 × v1 - v0 × r1

b. S := p/2 should be S := -p/2.

After these were made, the program ran smoothly for the following equations:

$$x^4 - 3x^3 + 20x^2 + 44x + 43 = 0 \quad x = -.97063897 \pm 1.0058076i$$

$$x = -2.4706390 \pm 4.6405330i$$

$$x^6 - 2x^5 + 2x^4 + x^3 + 6x^2 - 6x + 8 = 0$$

$$x = 0.50000000 \pm 0.86602539i$$

$$x = 1.00000000 \pm 1.0000000i$$

$$x = 1.50000000 \pm 1.3228756i$$

$$x^5 + x^4 - 8x^3 - 16x^2 + 7x + 15 = 0$$

$$x = .000000005, ** - 0.999999999$$

$$x = 3.00000000, 0.999999999$$

$$x = -2.00000000 \pm 1.0000000i$$

With the equation $x^5 + 7x^4 + 5x^3 + 6x^2 + 3x + 2 = 0$ convergence was slow, and full accuracy was not obtained. However, the

equation with reciprocal roots, $2x^5 + 3x^4 + 6x^3 + 5x^2 + 7x + 1 = 0$, converged rapidly.

* Work supported by the U. S. Atomic Energy Commission.

** Spurious zero real roots are introduced for equations of odd order.

CERTIFICATION OF ALGORITHM 3
SOLUTION OF POLYNOMIAL EQUATIONS BY
BAIRSTOW HITCHCOCK METHOD (A. A. Grau,
Comm. ACM, February, 1960)

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Bairstow was coded for the Burroughs 220 computer using the Burroughs ALGOL. Conversion from ALGOL 60 was made by hand on a statement-for-statement basis. The integer declaration had to be extended to include n, k, n, NAT, EX, and the corrections noted in the certification by Henry C. Thacher, Jr., *Communications ACM*, June, 1960, were incorporated.

By selecting the input parameters carefully, all branches of the routine were tested and the program ran smoothly. The following polynomial equations were solved:

$$\begin{aligned} x^6 - 14x^4 + 49x^2 - 36 = 0, & \quad x = \pm 1.0000000 \\ & \quad x = \pm 1.9999998 \\ & \quad x = \pm 3.0000001 \\ x^8 - 30x^6 + 273x^4 - 820x^2 + 576 = 0, & \quad x = \pm 1.0000000 \\ & \quad x = \pm 2.0000000 \\ & \quad x = \pm 2.9999999 \\ & \quad x = \pm 4.0000001 \end{aligned}$$

Several minor errors were found in the certification by Mr. Thacher. The constant term in the first polynomial should be 54 instead of 43, the second pair of roots for that polynomial should be $+ 2.470639 \pm 4.6405330 i$, and the second pair of roots for the second polynomial should be $-1.0 \pm i$.

REMARKS ON ALGORITHMS 2 AND 3 (*Comm. ACM*, February 1960), ALGORITHM 15 (*Comm. ACM*, August 1960) AND ALGORITHMS 25 AND 26 (*Comm. ACM*, November 1960)

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Algorithms 2, 15, 25 and 26 were all concerned with the calculation of zeros of arbitrary functions by successive linear or quadratic interpolation. The main limiting factor on the accuracy attainable with such procedures is the condition of the method of evaluating the function in the neighbourhood of the zeros. It is this condition which should determine the tolerance which is allowed for the relative error. With a well-conditioned method of evaluation quite a strict convergence criterion will be met, even when the function has multiple roots.

For example, a real quadratic root solver (of a type similar to Algorithm 25) has been used on ACE to find the zeros of triple-diagonal matrices T having $t_{ii} = a_i$, $t_{i+1,i} = b_{i+1}$, $t_{i,i+1} = c_{i+1}$. As an extreme case I took $a_1 = a_2 = \dots = a_5 = 0$, $a_6 = a_7 = \dots = a_{10} = 1$, $a_{11} = 2$, $b_1 = 1$, $c_1 = 0$ so that the function which was being evaluated was $x^5(x-1)^5(x-2)$. In spite of the multiplicity of the roots, the answers obtained using floating-point arithmetic with a 46-bit mantissa had errors no greater than 2^{-44} . Results of similar accuracy have been obtained for the same problem using linear interpolation in place of the quadratic. This is because the method of evaluation which was used, the two-

term recurrence relation for the leading principal minors, is a very well-conditioned method of evaluation. Knowing this, I was able to set a tolerance of 2^{-42} with confidence. If the same function had been evaluated from its explicit polynomial expansion, then a tolerance of about 2^{-7} would have been necessary and the multiple roots would have obtained with very low accuracy.

To find the zero roots it is necessary to have an absolute tolerance for $|x_{r+1} - x_r|$ as well as the relative tolerance condition. It is undesirable that the preliminary detection of a zero root should be necessary. The great power of rootfinders of this type is that, since we are not saddled with the problem of calculating the derivative, we have great freedom of choice in evaluating the function itself. This freedom is encroached upon if we frame the rootfinder so that it finds the zeros of $x = f(x)$ since the true function $x - f(x)$ is arbitrarily separated into two parts. The formal advantage of using this formulation is very slight. Thus, in Certification 2 (June 1960), the calculation of the zeros of $x = \tan x$ was attempted. If the function $(-x + \tan x)$ were used with a general zero finder then, provided the method of evaluation was, for example

$$\begin{aligned} x &= n\pi + y \\ \tan x - x &= -n\pi + \frac{y^3 - \frac{y^5}{30} - \dots}{\cos y}, \end{aligned}$$

the multiple zeros at $x = 0$ could be found as accurately as any of the others. With a slight modification of common sine and cosine routines, this could be evaluated as

$$-n\pi + \frac{(\sin y - y) - y(\cos y - 1)}{1 + (\cos y - 1)}$$

and the evaluation is then well-conditioned in the neighbourhood of $x = 0$. As regards the number of iterations needed, the restriction to 10 (Certification 2) is rather unreasonably small. For example, the direct evaluation of $x^{60} - 1$ is well conditioned, but starting with the values $x = 2$ and $x = 1.5$ a considerable number of iterations are needed to find the root $x = 1$. Similarly a very large number are needed for Newton's method, starting with $x = 2$. If the time for evaluating the derivative is about the same as that for evaluating the function (often it is much longer), then linear interpolation is usually faster, and quadratic interpolation much faster, than Newton.

In all of the algorithms, including that for Bairstow, it is useful to have some criterion which limits the permissible change from one value of the independent variable to the next [1]. This condition is met to some extent in Algorithm 25 by the condition S4, that $\text{abs}(fprt) < \text{abs}(x2 \times 10)$, but here the limitation is placed on the permissible increase in the value of the function from one step to the next. Algorithms 3 and 25 have tolerances on the size of the function and on the size of the remainders r1 and r0 respectively. They are very difficult tolerances to assign since these quantities may take very small values without our wishing to accept the value of x as a root. In Algorithm 3 (*Comm. ACM* June 1960) it is useful to return to the original polynomial and to iterate with each of the computed factors. This eliminates the loss of accuracy which may occur if the factors are not found in increasing order. This presumably was the case in Certification 3 when the roots of $x^5 + 7x^4 + 5x^3 + 6x^2 + 3x + 2 = 0$ were attempted. On ACE, however, all roots of this polynomial were found very accurately and convergence was very fast using single-precision, but the roots emerged in increasing order. The reference to slow convergence is puzzling. On ACE, convergence was fast for all the initial approximations to p and q which were tried. When the initial approximations used were such that the real root $x = -6.3509936103$ and the spurious zero were found first, the remaining two quadratic factors were of lower accuracy, though this was, of course, rectified by iteration in the original

polynomial. When either of the other two factors was found first, then all factors were fully accurate even without iteration in the original polynomial [1].

REFERENCE

- [1] J. H. WILKINSON. The evaluation of the zeros of ill-conditioned polynomials Parts I and II. *Num. Math.* 1 (1959), 150-180.

CERTIFICATION OF ALGORITHM 3
SOLUTION OF POLYNOMIAL EQUATION BY
BARSTOW-HITCHCOCK (A. A. Grau, *Comm. ACM*
Feb. 1960)

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Bairstow was transliterated into BALGOL and tested on the Burroughs 220. The corrections supplied by Thatcher, *Comm. ACM*, June 1960, were incorporated. Results were correct for equations for which the method is suitable. $x^4 - 16 = 0$ is one of those which gave nonsensical results. Seven-digit results were obtained for 12 test equations, one of which was $x^6 - 2x^5 + 2x^4 + x^3 + 6x^2 - 6x + 8 = 0$.