ALGORITHM 62
A SET OF ASSOCIATE LEGENDRE POLYNOMIALS OF THE SECOND KIND*

JOHN R. HERNDON
Stanford Research Institute, Menlo Park, California


```
comment This procedure places a set of values of Qn^m(x) in the
array Q[ ] for values of n from 0 to nmax for a particular value
of m and a value of x which is real if ri is 0 and is purely imaginary,
ix, ortherwise. R[ ] will contain the set of ratios of successive
values of Q. These ratios may be especially valuable when the
Qn^m(x) of the smallest size is so small as to underflow the machine
representation (e.g. 10^-60 if 10^-51 were the smallest representable
number). 9.9 × 10^45 is used to represent infinity. Imaginary
values of x may not be negative and real values of x may not be
smaller than 1.
  Values of Qn^m(x) may be calculated easily by hypergeometric
series if x is not too small nor (n - m) too large. Qn^m(x) can be
computed from an appropriate set of values of Pn^m(x) if x is near
1.0 or ix is near 0. Loss of significant digits occurs for x as small as
1.1 if n is larger than 10. Loss of significant digits is a major diffi-
culty in using finite polynomial representations also if n is larger
than m. However, QLEG has been tested in regions of x and n
both large and small;
procedure QLEG(m, nmax, x, ri, R, Q); value m, nmax, x, ri;
          real m, nmax, x, ri; real array R, Q;
begin   real t, i, n, q0, s;
        n := 20;
        if nmax > 13 then
            n := nmax + 7;
        if ri = 0 then
            begin if m = 0 then
                            Q[0] := 0.5 × log((x + 1)/(x - 1))
                            else
                      begin t := -1.0/sqrt(x × x - 1);
                            q0 := 0;
                            Q[0] := t;
                            for i := 1 step 1 until m do
                              begin s := (x+x)×(i-1)×t
                              ×Q[0]+(3i-i×i-2)×q0;
                              q0 := Q[0];
                              Q[0] := s end end;
            if      x = 1 then
                    Q[0] := 9.9 ↑ 45;
            R[n + 1] := x - sqrt(x × x - 1);
            for     i := n step -1 until 1 do
                    R[i] := (i + m)/((i + i + 1) × x
                    +(m - i - 1) × R[i + 1]);
            go to the end;
        if m = 0 then
            begin if x < 0.5 then
                    Q[0] := arctan(x) - 1.5707963 else
                    Q[0] := - arctan(1/x)end else
            begin   t := 1/sqrt(x × x + 1);
            q0 := 0;
            Q[0] := t;
            for     i := 2 step 1 until m do
                    begin s := (x + x) × (i - 1) × t × Q[0]
                    +(3i + i × i - 2) × q0;
                          q0 := Q[0];
                          Q[0] := s end end;
            R[n + 1] := x - sqrt(x × x + 1);
                for  i := n step - 1 until 1 do
                     R[i] := (i + m)/((i - m + 1) × R[i + 1]
                     -(i + i + 1) × x);
                for  i := 1 step 2 until nmax do
                     R[i] := - R[i];
        the: for i := 1 step 1 until nmax do
                     Q[i] := Q[i - 1] × R[i]
end QLEG;
```


* This procedure was developed in part under the sponsorship of the Air Force Cambridge Research Center.

REMARK ON ALGORITHM 62
A SET OF ASSOCIATE LEGENDRE POLYNOMIALS OF THE SECOND KIND (John R. Herndon, *Comm. ACM 4* (July, 1961))

JOHN R. HERNDON
Stanford Research Institute, Menlo Park, California

In regard to Algorithm 62 in *Communications of the ACM*, two errors were found:

The 14th line of the procedure

for i := 1 **step** 1 **until** m **do**

should read

for i := 2 **step** 1 **until** m **do**

The 35th line

+(3i − iXi − 2)Xq0

should read

+(3i − i × i − 2) × q0

The procedure *QLEC* was developed from the standard recurrence formula

$$(n + m - 1)Q^m_{n-2} = (2n - 1)\cdot x\cdot Q^m_{n-1} - (n - m)Q_n{}^m.$$

Invert and multiply by $(n + m - 1)Q^m_{n-1}$.

$$\frac{Q^m_{n-1}}{Q^m_{n-2}} = \frac{(n + m - 1)}{(2n - 1)\cdot x - (n - m)Q_n{}^m/Q^m_{n-1}},$$

or

$$R^m_{n-1} = \frac{(n + m - 1)}{(2n - 1)\cdot x - (n - m)R_n{}^m}.$$

Analysis (and testing) shows that, for n large, this infinite continued fraction need only be carried to about eight terms for eight-digit accuracy if the final term is evaluated with the asymptotic value derived by setting

$$R^m_{n-1} = R_n{}^m, \ \lim_{n\to\infty} R_n{}^m = x \pm \sqrt{x^2 - 1},$$

the minus sign being chosen since in general $Q_n{}^m < Q^m_{n-1}$. The formulas pertaining to purely imaginary parameters follow readily. The value of

$$Q_0{}^0(x) = \frac{1}{2}\log_e\frac{x + 1}{x - 1},$$

while

$$Q_1^0(x) = x \cdot Q_0^0(x) - 1,$$

and

$$Q_0^1(x) = \frac{-1}{\sqrt{x^2 - 1}}.$$

Other values are derived using the ratios $R_n^m(x)$ and/or the recurrence formula

$$Q_n^m = -\frac{2(m-1)x}{\sqrt{x^2-1}} Q_n^{m-1} + (n - m + 2)(n + m - 2)Q_n^{m-2}.$$

The derivation of the expression for $Q_0^0(ix)$ is not trivial and proceeds as follows:

$$i \cdot Q_0^0(ix) = \frac{1}{2} \log_e \frac{ix + 1}{ix - 1} = \frac{1}{2} \log_e \left[-\frac{x^2 - 1}{x^2 + 1} + \frac{2x}{x^2 + 1} \right]$$

$$e^{a+ib} = e^a \cdot e^{ib} = e^a \cos b + i \sin b.$$

Thus

$$\tan b = \frac{-2x}{1 - x^2}$$

and

$$Q_0^0(ix) = (\arctan x - \pi/2)i.$$